Sustitución trigonométrica

lunes, 25 de abril de 2022

Identidades pitagóricas:

$\boxed{\text{sen}^2 x + \cos^2 x = 1}$ $\boxed{\tan^2 x + 1 = \sec^2 x}$ $\cot^2 x + 1 = \csc^2 x$

Sea $a > 0$. * Siempre que se escriba a^2 consideraremos que $a > 0$, a menos que se indique lo contrario.

$\sqrt{a^2 - u^2} \longrightarrow u = a\,\text{sen}\,\theta$, con $\theta \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

$$\sqrt{a^2 - u^2} = \sqrt{a^2 - a^2\text{sen}^2\theta} = \sqrt{a^2(1 - \text{sen}^2\theta)} = \sqrt{a^2\cos^2\theta} = a\cos\theta$$

$\left(\cos\theta \geq 0 \text{ si } \theta \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]\right)$.

$\sqrt{a^2 + u^2} \longrightarrow u = a\tan\theta$, con $\theta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

$$\sqrt{a^2 + u^2} = \sqrt{a^2 + a^2\tan^2\theta} = \sqrt{a^2(1 + \tan^2\theta)} = a\sec\theta$$

$\left(\sec\theta > 0 \text{ si } \theta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)\right)$

$\sqrt{u^2 - a^2} \longrightarrow u = a\sec\theta$, con $\theta \in \left[0, \frac{\pi}{2}\right) \cup \left[\pi, \frac{3\pi}{2}\right)$.

$$\sqrt{u^2 - a^2} = \sqrt{a^2\sec^2\theta - a^2} = \sqrt{a^2(\sec^2\theta - 1)} = a\tan\theta$$

$\left(\tan\theta \geq 0 \text{ si } \theta \in \left[0, \frac{\pi}{2}\right) \cup \left[\pi, \frac{3\pi}{2}\right)\right)$.

*También podemos utilizar sustitución trigonométrica en integrales que incluyan $a^2 + x^2$, $a^2 - x^2$ o $x^2 - a^2$ sin raíces.

$\sqrt{a^2 - x^2}$

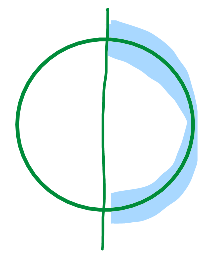
Si consideramos $a > 0$ y $x \in \mathbb{R}$ tales que $a^2 - x^2 \geq 0 \Rightarrow a^2 \geq x^2$

$\Rightarrow a \geq |x| \Rightarrow -a \leq x \leq a \Rightarrow -1 \leq \frac{x}{a} \leq 1.$

Dado que $\operatorname{sen} : \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \longrightarrow [-1, 1]$ es biyectiva y $\frac{x}{a} \in [-1, 1]$,

entonces existe un único $\theta \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ tal que $\operatorname{sen}\theta = \frac{x}{a}$

$\Rightarrow x = a \operatorname{sen}\theta$, con $\theta \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$



$\Rightarrow a \cos\theta \geq 0$

$\Rightarrow \sqrt{a^2 \cos^2\theta} = a \cos\theta.$

$\sqrt{a^2 + x^2}$

Si $a > 0$ y $x \in \mathbb{R}$, como $\tan\theta : \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \longrightarrow \mathbb{R}$ es biyectiva y $\frac{x}{a} \in \mathbb{R}$,

entonces existe $\theta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ tal que $\tan\theta = \frac{x}{a}$

$\Rightarrow x = a \tan\theta$, con $\theta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

$\Rightarrow a \sec\theta > 0$

$\Rightarrow \sqrt{a^2 \sec^2\theta} = a \sec\theta.$

$\sqrt{x^2 - a^2}$

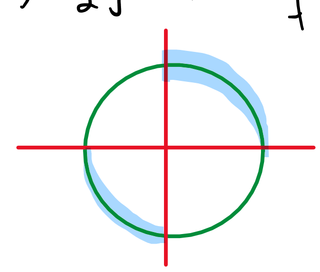
Si $a > 0$ y $x \in \mathbb{R}$ son tales que $x^2 - a^2 \geq 0$, entonces $x^2 \geq a^2$

$\Rightarrow |x| \geq a \Rightarrow x \leq -a$ o $a \leq x \Rightarrow \frac{x}{a} \leq -1$ o $1 \leq \frac{x}{a}$.

Dado que $\sec : \left[0, \frac{\pi}{2}\right) \cup \left[\pi, \frac{3\pi}{2}\right) \longrightarrow (-\infty, -1] \cup [1, \infty)$ es biyectiva,

entonces existe un único $\theta \in \left[0, \frac{\pi}{2}\right) \cup \left[\pi, \frac{3\pi}{2}\right]$ tal que $\sec\theta = \frac{x}{a}$

$\Rightarrow x = a \sec\theta$



$\Rightarrow a \tan\theta \geq 0$

$\Rightarrow \sqrt{a^2 \tan^2\theta} = a \tan\theta.$

Recordatorio:

$$\frac{d}{dx} \operatorname{arc\,sen} x = \frac{1}{\sqrt{1-x^2}} \qquad \frac{d}{dx} \operatorname{arc\,cos} x = \frac{-1}{\sqrt{1-x^2}}$$

$$\frac{d}{dx} \operatorname{arc\,tan} x = \frac{1}{1+x^2} \qquad \frac{d}{dx} \operatorname{arc\,cot} x = \frac{-1}{1+x^2}$$

$$\frac{d}{dx} \operatorname{arc\,sec} x = \frac{1}{x\sqrt{x^2-1}} \qquad \frac{d}{dx} \operatorname{arc\,csc} x = \frac{-1}{x\sqrt{x^2-a^2}}$$

$$\int \frac{dx}{\sqrt{1-x^2}} = \operatorname{arc\,sen} x + C.$$

$$\int \frac{-1}{\sqrt{1-x^2}}\,dx = -\int \frac{1}{\sqrt{1-x^2}}\,dx = -\operatorname{arc\,sen} x + C$$

$$\operatorname{arc\,cos} x + C$$

$\boxed{a > 0}$

$$\int \frac{dx}{\sqrt{a^2-x^2}} = \int \frac{a\cos\theta\,d\theta}{a\cos\theta} = \int d\theta = \theta + c$$

$$= \boxed{\operatorname{arc\,sen}\left(\frac{x}{a}\right) + c}$$

$x = a \operatorname{sen} \theta$

$dx = a\cos\theta\,d\theta$

$\frac{x}{a} = \operatorname{sen}\theta \rightarrow \theta = \operatorname{arc\,sen}\left(\frac{x}{a}\right)$

$$\int \frac{dx}{a^2+x^2} = \int \frac{a\sec^2\theta\, d\theta}{a^2\sec^2\theta} = \int \frac{1}{a}\, d\theta = \frac{1}{a}\theta + c$$

$x = a\tan\theta$

$dx = a\sec^2\theta\, d\theta$

$\theta = \arctan\left(\frac{x}{a}\right)$

$$= \boxed{\frac{1}{a}\arctan\left(\frac{x}{a}\right) + c}$$

$$a^2 + x^2 = a^2 + a^2\tan^2\theta = a^2(1+\tan^2\theta) = a^2\sec^2\theta$$

$$\int \frac{dx}{x\sqrt{x^2-a^2}} = \int \frac{a\cancel{\sec\theta}\,\cancel{\tan\theta}}{a\cancel{\sec\theta}\,(a\cancel{\tan\theta})}\, d\theta = \int \frac{1}{a}\, d\theta = \frac{1}{a}\theta + c$$

$x = a\sec\theta$

$dx = a\sec\theta\tan\theta\, d\theta$

$\theta = \operatorname{arc\,sec}\left(\frac{x}{a}\right)$

$$= \boxed{\frac{1}{a}\operatorname{arc\,sec}\left(\frac{x}{a}\right) + c}$$

Integrales de Funciones trigonométricas:

$$\int \operatorname{sen} x\, dx = -\cos x + c$$

$$\int \cos x\, dx = \operatorname{sen} x + c$$

$$\log\left(\frac{1}{|\cos x|}\right) = -\log|\cos x|$$

$$\int \tan x\, dx = \log|\sec x| + c = -\log|\cos x| + c$$

$$\int \cot x\, dx = \log|\operatorname{sen} x| + c$$

$$\int \sec x\, dx = \log|\sec x + \tan x| + c$$

$$\int \csc x\, dx = \log|\csc x - \cot x| + c = -\log|\csc x + \cot x| + c$$

$$\int \frac{x}{\sqrt{25-x^2}}\,dx = \int \frac{5\ \text{sen}\,\theta\,(\cancel{5\cos\theta})}{\cancel{5\cos}\,\theta}\,d\theta = \int 5\ \text{sen}\,\theta\,d\theta$$

$x = 5\ \text{sen}\,\theta$

$dx = 5\cos\theta\,d\theta$

$\theta = \text{arc sen}\left(\frac{x}{5}\right)$

$$= -5\cos\theta + c = -5\cos\left(\text{arc sen}\left(\tfrac{x}{5}\right)\right) + c$$

$$= -5\sqrt{1-\text{sen}^2\left(\text{arc sen}\left(\tfrac{x}{5}\right)\right)} + c$$

$$= -5\sqrt{1-\left(\tfrac{x}{5}\right)^2} + c = -5\sqrt{1-\frac{x^2}{25}} + c$$

$$= -5\sqrt{\frac{25-x^2}{25}} + c = -5\left(\frac{\sqrt{25-x^2}}{5}\right) + c$$

$$= \boxed{-\sqrt{25-x^2} + c}$$

$\cos x = \sqrt{1-\text{sen}^2 x}$ si $\cos x \geq 0$

$\text{sen}^2 x = (\text{sen}\ x)^2$ $\qquad \text{sen}^2(\text{arc sen}\ x) = (\text{sen}(\text{arc sen}\ x))^2 = x^2$

Otras formas de resolver la integral pasada:

$$\int \frac{x}{\sqrt{25-x^2}}\,dx = \int \frac{5\ \text{sen}\,\theta\,(\cancel{5\cos\theta})}{\cancel{5\cos}\,\theta}\,d\theta = \int 5\ \text{sen}\,\theta\,d\theta$$

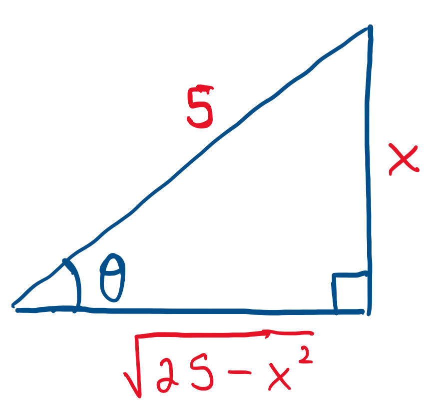
$x = 5\ \text{sen}\,\theta$

$dx = 5\cos\theta\,d\theta$

$\text{sen}\,\theta = \frac{x}{5} = \frac{C.O.}{H}$

$$= -5\cos\theta + c$$

$$= -5\left(\frac{\sqrt{25-x^2}}{5}\right) + c = \boxed{-\sqrt{25-x^2} + c}$$



$\cos\theta = \dfrac{\sqrt{25-x^2}}{5} \leftarrow \dfrac{C.A.}{H}$

$$\int \frac{x}{\sqrt{25-x^2}}\,dx = -\frac{1}{2}\int \frac{-2x}{\sqrt{25-x^2}}\,dx = -\frac{1}{2}\int \frac{du}{\sqrt{u}} = -\frac{1}{2}\int u^{-\frac{1}{2}}\,du$$

$u = 25 - x^2$

$du = -2x\,dx$

$$= -\frac{1}{2}\left(2u^{\frac{1}{2}}\right) + c$$

$$= -u^{1/2} + c = \boxed{-\sqrt{25-x^2} + c}$$

$$\int u^{-\frac{1}{2}}\,du = \frac{u^{1/2}}{\frac{1}{2}} + c$$

$$\int \frac{dx}{\sqrt{x^2-a^2}} = \int \frac{a\sec\theta\tan\theta\,d\theta}{a\tan\theta} = \int \sec\theta\,d\theta$$

$x = a\sec\theta$

$dx = a\sec\theta\tan\theta\,d\theta$

$\sec\theta = \frac{x}{a} = \frac{H}{C.A.}$

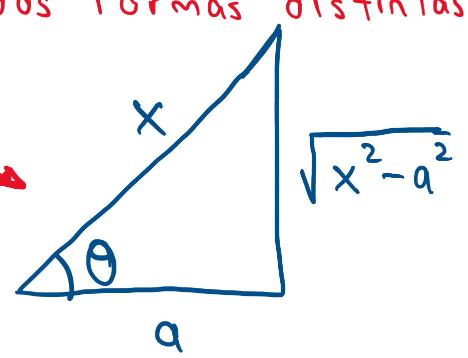
$$= \log\left|\sec\theta + \tan\theta\right| + c$$

$$= \boxed{\log\left|\frac{x}{a} + \frac{\sqrt{x^2-a^2}}{a}\right| + c}$$

$$= \boxed{\log\left|\frac{x+\sqrt{x^2-a^2}}{a}\right| + c}$$

$\tan\theta = \sqrt{\sec^2\theta - 1} = \sqrt{\left(\frac{x}{a}\right)^2 - 1} = \frac{\sqrt{x^2-a^2}}{a}$

(dos Formas distintas de calcular $\tan\theta$)



$\tan\theta = \frac{\sqrt{x^2-a^2}}{a}$

$\frac{d}{dx}\tan x = \sec^2 x$

$\frac{d}{dx}\cot x = -\csc^2 x$

$\int \sec^2 x\,dx = \tan x + c$

$\int \csc^2 x\,dx = -\cot x + c$

$$\int \frac{\sqrt{a^2-x^2}}{x^2}\,dx = \int \frac{a\cos\theta\,(a\cos\theta)\,d\theta}{a^2\,\text{sen}^2\theta} = \int \frac{\cancel{a^2}\cos^2\theta}{\cancel{a^2}\,\text{sen}^2\theta}\,d\theta$$

$x = a\,\text{sen}\,\theta$

$dx = a\cos\theta\,d\theta$

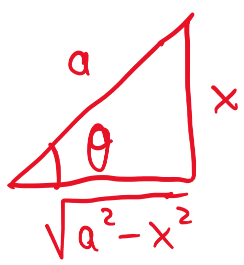
$1+\cot^2\theta = \csc^2\theta$

$\text{sen}\,\theta = \frac{x}{a}$

$$= \int \cot^2\theta\,d\theta = \int \csc^2\theta - 1\,d\theta$$

$$= -\cot\theta - \theta + C$$

$$= \boxed{-\frac{\sqrt{a^2-x^2}}{x} - \text{arc sen}\left(\frac{x}{a}\right) + C}$$



$\cot\theta = \frac{\sqrt{a^2-x^2}}{x}$

Otra Forma:

$$\int \frac{\cancel{a^2}\cos^2\theta}{\cancel{a^2}\,\text{sen}^2\theta}\,d\theta = \int \frac{1-\text{sen}^2\theta}{\text{sen}^2\theta}\,d\theta = \int \frac{1}{\text{sen}^2\theta} - 1\,d\theta$$

$$= \int \csc^2\theta - 1\,d\theta = \cdots$$

$$\int \tan^2 x\,dx = \int \sec^2 x - 1\,dx = \tan x - x + C$$

$$\int \frac{dx}{4x^2+49} = \int \frac{\frac{7}{2}\cancel{\sec^2\theta}}{49\cancel{\sec^2\theta}}\,d\theta = \frac{1}{14}\int d\theta = \frac{1}{14}\theta + C$$

$$= \boxed{\frac{1}{14}\arctan\left(\frac{2x}{7}\right) + C}$$

$(2x)^2 + 49$

$u = 2x \qquad a = 7$

$2x = 7\tan\theta \rightarrow \frac{2x}{7} = \tan\theta$

$\theta = \arctan\left(\frac{2x}{7}\right)$

$x = \frac{7}{2}\tan\theta$

$dx = \frac{7}{2}\sec^2\theta\, d\theta$

$4x^2 + 49 = 49\tan^2\theta + 49 = 49(\tan^2\theta + 1)$

$= 49\sec^2\theta$

$x^2 + a^2 \longrightarrow x = a\tan\theta$

$u^2 + a^2 \longrightarrow u = a\tan\theta$