viernes, 22 de abril de 2022

- Sustitución directa $u = e^x + 1$

$$\int \frac{e^{2x}}{\sqrt{e^x+1}}\,dx = \int \frac{e^x \cdot e^x}{\sqrt{e^x+1}}\,dx = \int \frac{u-1}{\sqrt{u}}\,du = \int \frac{u}{\sqrt{u}} - \frac{1}{\sqrt{u}}\,du$$

$u = e^x + 1$

$du = e^x\,dx$

$u - 1 = e^x$

$$= \int u^{\frac{1}{2}} - u^{-\frac{1}{2}}\,du = \frac{2u^{\frac{3}{2}}}{3} - 2u^{\frac{1}{2}} + c$$

$$= \boxed{\frac{2}{3}(e^x+1)^{3/2} - 2(e^x+1)^{\frac{1}{2}} + C}$$

$e^x = e^x + 1 - 1$

$= u - 1$

- Sustitución inversa $u = e^x + 1$

$$\int \frac{e^{2x}}{\sqrt{e^x+1}}\,dx = \int \frac{(u-1)^{\cancel{2}}}{\sqrt{u}} \cdot \frac{1}{\cancel{u-1}}\,du = \int \frac{u-1}{\sqrt{u}}\,du = \cdots$$

$u = e^x + 1$

$u - 1 = e^x \rightarrow (u-1)^2 = e^{2x}$

$\downarrow$

$\log(u-1) = x$

$\frac{1}{u-1}\,du = dx$

• Sustitución directa $u=\sqrt{e^x+1}$

$$\int \frac{e^{2x}}{\sqrt{e^x+1}}\,dx = 2\int \frac{e^x \cdot e^x}{2\sqrt{e^x+1}}\,dx = 2\int u^2-1\,du = \cdots$$

$$u=\sqrt{e^x+1}$$

$$du = \frac{e^x}{2\sqrt{e^x+1}}\,dx$$

$$u^2-1 = e^x \qquad e^x = \left(\sqrt{e^x+1}^{\,2}-1\right)$$

Sustitución inversa $u=\sqrt{e^x+1}$

$$\int \frac{e^{2x}}{\sqrt{e^x+1}}\,dx = \int \frac{(u^2-1)^{\cancel{2}}}{\cancel{u}}\cdot\frac{2\cancel{u}}{\cancel{(u^2-1)}}\,du = 2\int u^2-1\,du$$

$$u=\sqrt{e^x+1}$$

$$u^2 = e^x+1$$

$$u^2-1 = e^x \rightarrow (u^2-1)^2 = e^{2x}$$

$$\log(u^2-1) = x$$

$$\frac{2u}{u^2-1}\,du = dx$$

$$= 2\left(\frac{u^3}{3}-u\right)+C$$

$$= \boxed{\frac{2}{3}\left(\sqrt{e^x+1}\right)^3 - 2\sqrt{e^x+1} + C}$$

¿En qué consiste la sustitución inversa?

Sustitución

directa

$u = g(x)$

$du = g'(x)\,dx$

(Sustituimos $g'(x)\,dx$ por du)

inversa

$u = g(x)$ despejar x

$g^{-1}(u) = x$

$(g^{-1})'(u)\,du = dx$

(Sustituir dx en la integral)

$$\int e^{\sqrt{x}}\,dx = \int e^{\sqrt{x}} \cdot (2\sqrt{x}) \cdot \frac{1}{2\sqrt{x}}\,dx = \int e^{u}(2u)\,du$$

$u = \sqrt{x}$

$du = \frac{1}{2\sqrt{x}}\,dx$

$\frac{1}{g'(x)}$ $\qquad g'(x)$

$x = u^2$

¿Siempre podemos utilizar sustitución inversa? <u>Sí</u>

Sustitución directa

$$\int F(g(x))\,dx = \int F(g(x)) \cdot \frac{1}{g'(x)} \cdot g'(x)\,dx \quad (\text{si } g'(x) \neq 0)$$

$u = g(x)$

$du = g'(x)\,dx$

$$= \boxed{\int F(u) \cdot \frac{1}{g'(g^{-1}(u))}\,du}$$

$g^{-1}(u) = x$ (si g es invertible)

Sustitución inversa

$$\int F(g(x))\,dx = \boxed{\int F(u)\cdot (g^{-1})'(u)\,du}$$

$u = g(x)$

$g^{-1}(u) = x$ (g invertible)

$(g^{-1})'(u)\,du = dx$

Como $(g^{-1})'(u) = \dfrac{1}{g'(g^{-1}(u))}$ siempre que $g'(g^{-1}(u)) \neq 0$, concluimos que las integrales obtenidas son iguales.

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* Si g' es continua y $g'(x) \neq 0$ para toda x en un intervalo I, por el T.V.I. sabemos que g' siempre es positiva o g' siempre es negativa $\Rightarrow$ g es estrictamente creciente en I o g es estrictamente decreciente en I $\Rightarrow$ g es inyectiva.

$$\int \sec x\,dx = \int \frac{\sec x\,(\sec x + \tan x)}{\sec x + \tan x}\,dx = \int \frac{\sec^2 x + \sec x \tan x}{\sec x + \tan x}\,dx$$

$u = \sec x + \tan x$

$du = \sec x \tan x + \sec^2 x\,dx$

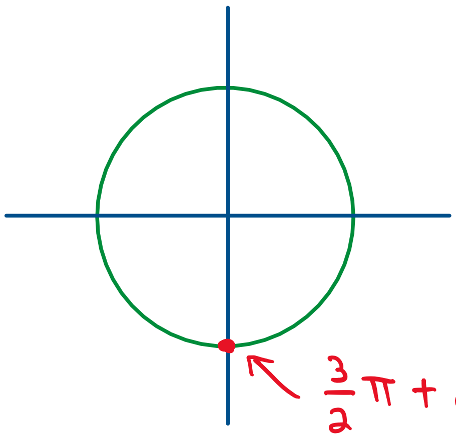
$$= \int \frac{du}{u} = \log|u| + C$$

$$= \boxed{\log|\sec x + \tan x| + C}$$

si $\sec x + \tan x \neq 0$

$$* \quad \sec x + \tan x = 0 \iff \frac{1}{\cos x} + \frac{\operatorname{sen} x}{\cos x} = 0 \iff \frac{1+\operatorname{sen} x}{\cos x} = 0$$

$$\iff 1 + \operatorname{sen} x = 0 \iff \operatorname{sen} x = -1$$



$\frac{3}{2}\pi + 2k\pi$

$$\iff x = \frac{3}{2}\pi + 2k\pi, \quad \text{con } k \in \mathbb{Z}$$

$$\implies \cos x = 0$$

$$\implies \sec x = \frac{1}{\cos x} \text{ no está definido.}$$

$\therefore \quad \sec x + \tan x \neq 0$ para toda $x \in \operatorname{Dom}(\sec)$

$\int \sec x \, dx = \log|\sec x + \tan x| + c$ para toda x en el dominio de sec.

$$\int \csc x \, dx = \int \frac{\csc x(\csc x + \cot x)}{\csc x + \cot x} dx = -\int \frac{-(\csc^2 x + \csc x \cot x)}{\csc x + \cot x} dx$$

$u = \csc x + \cot x$

$du = -\csc x \cot x - \csc^2 x \, dx$

$\frac{d}{dx} \csc x = -\csc x \cot x$

$\frac{d}{dx} \cot x = -\csc^2 x$

$$= -\int \frac{du}{u} = -\log|u| + c$$

$$= \boxed{-\log|\csc x + \cot x| + c}$$

si $\csc x + \cot x \neq 0$

Sustitución trigonométrica

$\boxed{\operatorname{sen}^2 x + \cos^2 x = 1}$ $\boxed{\tan^2 x + 1 = \sec^2 x}$ $\cot^2 x + 1 = \csc^2 x$

$a \neq 0$ (podemos suponer que $a > 0$).

$\sqrt{a^2 - x^2} \longrightarrow x = a \operatorname{sen}\theta$ (con $0 \leq \theta \leq \frac{\pi}{2}$)

$$\sqrt{a^2 - x^2} = \sqrt{a^2 - a^2 \operatorname{sen}^2\theta} = \sqrt{a^2(1 - \operatorname{sen}^2\theta)} = \sqrt{a^2\cos^2\theta} = a\cos\theta$$

$\sqrt{a^2 + x^2} \longrightarrow x = a\tan\theta$

$$\sqrt{a^2 + x^2} = \sqrt{a^2 + a^2\tan^2\theta} = \sqrt{a^2(1 + \tan^2\theta)} = a\sec\theta$$

$\sqrt{x^2 - a^2} \longrightarrow x = a\sec\theta$

$$\sqrt{x^2 - a^2} = \sqrt{a^2\sec^2\theta - a^2} = \sqrt{a^2(\sec^2\theta - 1)} = a\tan\theta$$

Observaciones:

* Si $\theta \in [0, \frac{\pi}{2}] \Rightarrow \begin{array}{l} \operatorname{sen}\theta \geq 0 \\ \cos\theta \geq 0 \end{array}$

$\sqrt{a^2 - x^2}$

Si consideramos $a > 0$ y $x \geq 0$, como $a^2 - x^2 \geq 0 \Rightarrow a^2 \geq x^2 \geq 0$

$\Rightarrow a \geq x \geq 0$

$\Rightarrow 1 \geq \frac{x}{a} \geq 0$

Como $\operatorname{sen}: [0, \frac{\pi}{2}] \rightarrow [0, 1]$ es biyectiva y $\frac{x}{a} \in [0, 1]$, entonces existe un único $\theta \in [0, \frac{\pi}{2}]$ tal que

$$\operatorname{sen}\theta = \frac{x}{a}$$

$$\Rightarrow x = a\operatorname{sen}\theta, \quad \text{con } \theta \in [0, \tfrac{\pi}{2}].$$

De manera similar, podemos definir $x = a\tan\theta$, con $\theta \in [0, \frac{\pi}{2}]$, en el segundo caso o $x = \sec\theta$, con $\theta \in [0, \frac{\pi}{2}]$ para el tercer caso.

$$\int \frac{dx}{\sqrt{7 - x^2}} = \int \frac{\sqrt{7}\cos\theta \, d\theta}{\sqrt{7}\cos\theta} = \int d\theta = \theta + C = \operatorname{arc\,sen}\left(\frac{x}{\sqrt{7}}\right) + C$$

$a = \sqrt{7}$

$$x = \sqrt{7}\,\operatorname{sen}\theta$$

$$dx = \sqrt{7}\cos\theta \, d\theta$$

$$\frac{x}{\sqrt{7}} = \operatorname{sen}\theta$$

$$\theta = \operatorname{arc\,sen}\left(\frac{x}{\sqrt{7}}\right)$$

En general:

$$\int \frac{dx}{\sqrt{a^2 - x^2}} = \operatorname{arc\,sen}\left(\frac{x}{a}\right) + C$$