Ejercicios de derivadas de funciones

martes, 11 de enero de 2022

$$\boxed{f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}}$$

Ejercicio. Demuestra que si $f(x) = \sqrt{x}$ y $a > 0$, entonces $f'(a) = \dfrac{1}{2\sqrt{a}}$.

Sol.

$$\lim_{h \to 0} \frac{f(a+h) - f(a)}{h} = \lim_{h \to 0} \frac{\sqrt{a+h} - \sqrt{a}}{h} \cdot 1$$

$0 < |h| < \delta$

$\Rightarrow h \neq 0$

$$= \lim_{h \to 0} \frac{\sqrt{a+h} - \sqrt{a}}{h} \cdot \frac{\sqrt{a+h} + \sqrt{a}}{\sqrt{a+h} + \sqrt{a}}$$

$$= \lim_{h \to 0} \frac{(a+h) - a}{h(\sqrt{a+h} + \sqrt{a})}$$

$$= \lim_{h \to 0} \frac{\cancel{h}}{\cancel{h}(\sqrt{a+h} + \sqrt{a})} \qquad h \neq 0$$

$$= \lim_{h \to 0} \frac{1}{\underline{\sqrt{a+h} + \sqrt{a}}} \quad \text{continua en } h = 0$$

$$= \frac{1}{\sqrt{a+0} + \sqrt{a}}$$

$$\therefore f'(a) = \frac{1}{2\sqrt{a}} \quad \blacksquare$$

Ejercicio. Demuestra usando la definición que si $f(x) = \dfrac{1}{x}$ entonces $f'(a) = -\dfrac{1}{a^2}$ para $a \neq 0$, y que la recta tangente a G_f en $(a, f(a))$ solo corta a G_f en $(a, f(a))$.

Sol. Por definición

$$\lim_{h \to 0} \frac{f(a+h) - f(a)}{h} = \lim_{h \to 0} \frac{\frac{1}{a+h} - \frac{1}{a}}{h}$$

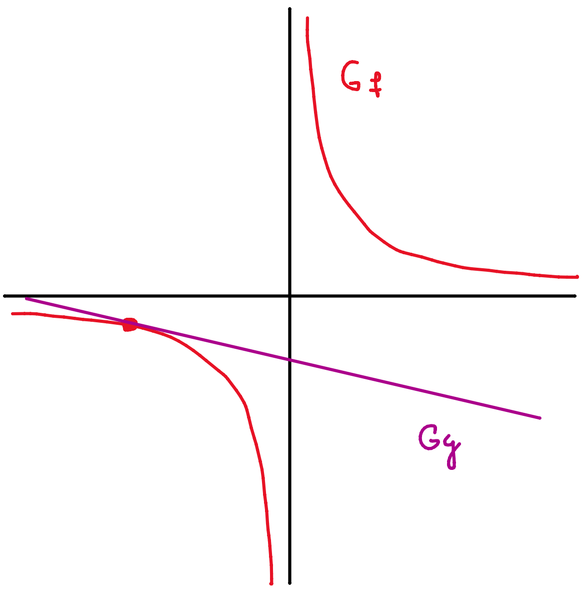
$$= \lim_{h\to 0} \frac{\frac{a-(a+h)}{(a+h)a}}{h}$$

$$= \lim_{h\to 0} \frac{-\cancel{h}}{\cancel{h}(a+h)a} \qquad h\neq 0$$

$$= \lim_{h\to 0} \frac{-1}{\underbrace{a^2+ha}} \qquad \text{función continua en } h=0$$

$$= \frac{-1}{a^2+0\cdot a}$$

$$\therefore\ f'(a) = -\frac{1}{a^2}$$



La ecuación de la recta tangente a G_f por $(a, f(a))$ esta dada por la función

$$g(x) = f'(a)(x-a) + f(a)$$

$$= -\frac{1}{a^2}(x-a) + \frac{1}{a}$$

$$= -\frac{x}{a^2} + \frac{1}{a} + \frac{1}{a}$$

$$\Rightarrow \boxed{g(x) = \frac{2}{a} - \frac{x}{a^2}}$$

Consideremos un punto en la intersección de G_f con la recta G_g,

es decir, sea $P \in G_f \cap G_g$

$\Rightarrow \quad P \in G_f \quad y \quad P \in G_g$

$\Rightarrow \quad P = (b, f(b)) \quad$ p.a. $b \neq 0 \quad y \quad P = (x, g(x)) \quad$ p. a. $x \in \mathbb{R}$

$\Rightarrow \quad (b, f(b)) = (x, g(x))$

$\Rightarrow \quad b = x \quad y \quad f(b) = g(x)$

$\Rightarrow \quad b = x \quad y \quad \frac{1}{b} = \frac{2}{a} - \frac{x}{a^2}$

$$\Rightarrow \quad \frac{1}{b} = \frac{2}{a} - \frac{b}{a^2} \qquad \times\, a^2 b \neq 0$$

$$\Rightarrow \quad a^2 = 2ab - b^2$$

$$\Rightarrow \quad (a-b)^2 = a^2 - 2ab + b^2 = 0$$

$$\Rightarrow \quad a - b = 0$$

$$\Rightarrow \quad a = b$$

$\therefore$ $P = (a, f(a))$, es decir, la recta G_g solo interseca a G_f en $(a, f(a))$ $\blacksquare$

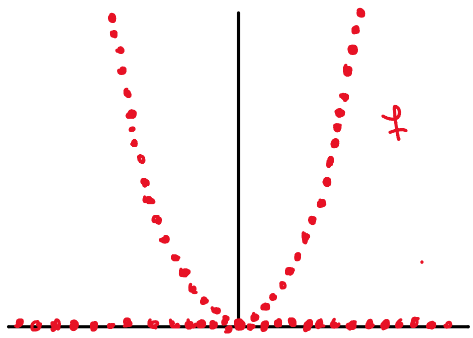
Ejercicio. Sea $f: \mathbb{R} \to \mathbb{R}$ definida como

$$f(x) = \begin{cases} x^2 & \text{si } x \in \mathbb{Q} \\ 0 & \text{si } x \in \mathbb{R} \setminus \mathbb{Q} \end{cases}$$

Demuestra que f es derivable en $x = 0$.

Sol. Como $0 \in \mathbb{Q}$ $\Rightarrow$ $f(0) = 0^2 = 0$

$$\Rightarrow \lim_{h \to 0} \frac{f(0+h) - f(0)}{h} = \lim_{h \to 0} \frac{f(h)}{h}$$



P.D. $\displaystyle\lim_{h \to 0} \frac{f(h)}{h} = 0$.

Sea $h \neq 0$.

Si $h \in \mathbb{Q}$ $\Rightarrow$ $f(h) = h^2 \geq 0$

Si $h \in \mathbb{R} \setminus \mathbb{Q}$ $\Rightarrow$ $f(h) = 0 \leq 0$

$$\Rightarrow \quad 0 \leq f(h) \leq h^2 \qquad \forall\, h \neq 0$$

$$\Rightarrow \quad -h^2 \leq f(h) \leq h^2$$

$h \neq 0$

$$\Rightarrow \quad -h \leq \frac{f(h)}{h} \leq h$$

$$\Rightarrow \quad 0 = \lim_{h\to 0} -h \leq \lim_{h\to 0} \frac{f(h)}{h} \leq \lim_{h \to 0} h = 0$$

$$\Rightarrow \quad \lim_{h\to 0} \frac{f(h)}{h} = 0$$

$\therefore$ f es derivable en 0 y $f'(0) = 0$ ∎

- $(f+g)'(x) = f'(x) + g'(x)$
- $(fg)'(x) = f(x)g'(x) + f'(x)g(x)$
- $\left(\frac{f}{g}\right)'(x) = \frac{f'(x)g(x) - f(x)g'(x)}{[g(x)]^2}$
- $c \in \mathbb{R} \Rightarrow (c)' = 0$
- $(cf)'(x) = c f'(x)$
- $(f \circ g)(x) = f'(g(x)) \cdot g'(x)$

- $(x^n)' = n x^{n-1}$
- $(\operatorname{sen} x)' = \cos x$
- $(\cos x)' = -\operatorname{sen} x$

Ejercicio. Calcula la derivada de las siguientes funciones:

1) $f(x) = \cos[(x+1)^2(x+2)]$

$g(x) = \cos x$ y $h(x) = (x+1)^2(x+2)$

$$\begin{aligned}
\Rightarrow f'(x) &= (g \circ h)'(x) \\
&= g'(h(x)) \cdot h'(x) \\
&= -\operatorname{sen}[(x+1)^2(x+2)] \cdot [(x+1)^2(x+2)]' \quad (f \cdot g)' \\
&= -\operatorname{sen}[(x+1)^2(x+2)] \cdot [(x+1)^2(x+2)' + ((x+1)^2)'(x+2)] \\
&= -\operatorname{sen}[(x+1)^2(x+2)] \cdot [(x+1)^2 \cdot 1 + 2(x+1) \cdot 1\ (x+2)]
\end{aligned}$$

$$= -\operatorname{sen}[(x+1)^2(x+2)]\cdot[(x+1)^2(x+2)' + ((x+1)^2)'(x+2)]$$

$$= -\operatorname{sen}[(x+1)^2(x+2)]\cdot[(x+1)^2\cdot 1 + 2(x+1)\cdot 1\,(x+2)]$$

$$= -\operatorname{sen}[(x+1)^2(x+2)]\cdot[(x+1)^2 + 2(x+1)(x+2)] \checkmark$$

$$= -(x+1)[(x+1)+2(x+2)]\operatorname{sen}[(x+1)^2(x+2)]$$

$$= -(x+1)(3x+5)\operatorname{sen}[(x+1)^2(x+2)]$$

2) $f(x) = \cos^3(x^2+\cos x)$

$$f'(x) = [(\cos(x^2+\cos x))^3]' \searrow (g\circ h)' \qquad g(x)=x^3 \text{ y } h(x)=\cos(\ldots)$$

$$= 3(\cos(x^2+\cos x))^2\,[\cos(x^2+\cos x)]'$$

regla cadena

$$= 3\cos^2(x^2+\cos x)\,(-\operatorname{sen}(x^2+\cos x))\,[x^2+\cos x]'$$

↓ suma

$$= -3\cos^2(x^2+\cos x)\cdot\operatorname{sen}(x^2+\cos x)\,[(x^2)'+(\cos x)']$$

$$= -3\cos^2(x^2+\cos x)\cdot\operatorname{sen}(x^2+\cos x)\,[2x-\operatorname{sen} x]$$

$$= 3(\operatorname{sen} x - 2x)\cdot\cos^2(x^2+\cos x)\cdot\operatorname{sen}(x^2+\cos x)$$

3) $f(x) = \cos^2[(x+\cos x)^2]$

$$f'(x) = [(\cos((x+\cos)^2))^2]'$$

$$= 2\cos((x+\cos x)^2)\,[\cos((x+\cos x)^2)]'$$

$$= 2\cos((x+\cos x)^2)\,(-\operatorname{sen}((x+\cos x)^2))\,[(x+\cos x)^2]'$$

$$= -2\cos((x+\cos x)^2)\operatorname{sen}((x+\cos x)^2)\cdot 2(x+\cos x)\,[x+\cos x]'$$

$$= -4\cos((x+\cos)^2)\operatorname{sen}((x+\cos x)^2)\cdot(x+\cos x)(1-\operatorname{sen} x)$$

4) $f(x) = \cos\left(\dfrac{x^3}{\operatorname{sen} x^3}\right)$

$$f'(x) = \left[\cos\left(\frac{x^3}{\operatorname{sen} x^3}\right)\right]'$$

$$= -\text{sen}\left(\frac{x^3}{\text{sen}\, x^3}\right)\left[\frac{x^3}{\text{sen}\, x^3}\right]'$$

$$= -\text{sen}\left(\frac{x^3}{\text{sen}\, x^3}\right)\left[\frac{(x^3)'\,\text{sen}(x^3) - x^3\,(\text{sen}(x^3))'}{(\text{sen}(x^3))^2}\right]$$

$$= -\text{sen}\left(\frac{x^3}{\text{sen}\, x^3}\right)\left[\frac{3x^2\cdot \text{sen}\, x^3 - x^3\cdot \cos(x^3)\cdot 3x^2}{\text{sen}^2(x^3)}\right]$$

$$= -3x^2\cdot \text{sen}\left(\frac{x^3}{\text{sen}\, x^3}\right)\cdot \frac{\text{sen}\, x^3 - x^3\cdot \cos x^3}{\text{sen}^2(x^3)}$$

5) $f(x) = \cos^3(\cos^2(\cos x))$

$$f'(x) = \left[(\cos(\cos^2(\cos x)))^3\right]'$$

$$= 3(\cos(\cos^2(\cos x)))^2\cdot\left[\cos(\cos^2(\cos x))\right]'$$

$$= 3\cos^2(\cos^2(\cos x))\cdot(-\text{sen}(\cos^2(\cos x)))\cdot\left[\cos^2(\cos x)\right]'$$

$$= -3\cos^2(\cos^2(\cos x))\cdot \text{sen}(\cos^2(\cos x))\cdot 2\cos(\cos x)\cdot\left[\cos(\cos x)\right]'$$

$$= \cancel{-}6\cos^2(\cos^2(\cos x))\,\text{sen}(\cos^2(\cos x))\,\cos(\cos x)\,(\cancel{-}\text{sen}(\cos x))\cdot\left[\cos x\right]'$$

$$= -6\cos^2(\cos^2(\cos x))\cdot \text{sen}(\cos^2(\cos x))\cdot \cos(\cos x)\cdot \text{sen}(\cos x)\cdot \text{sen}\, x$$